J. Fluid Mech. (1965), *vol.* 21, *part* 4, *pp.* 723–736
Printed in Great Britain

Electrokinetic flow round an elliptic cylinder in a finite fluid

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(Received 10 June 1963 and in revised form 10 March 1964)

The flow round an elliptic cylinder immersed in a finite volume of fluid and influenced by an electric field is determined, under the boundary conditions arising from electrosmosis at the surface of the cylinder. This is a problem in slow viscous flow with difficult boundary conditions, and is solved by numerical analysis, using the relaxation method. After the flow pattern has been determined, the mechanical action on the cylinder is calculated. An experimental application based on the theoretical results is suggested as a method for the determination of electrokinetic potentials.

1. Introduction

We consider an elliptic cylinder immersed with its generators vertical in a tank of horizontal square cross-section. The vertical walls of the tank are electrically conducting and are insulated from each other. The cylinder and fluid are either both conducting or both non-conducting, and we suppose that an electric field is applied between two opposite walls of the tank. Clearly, far from the top and bottom of the system we have, approximately, a two-dimensional electrostatic or steady-current problem in which, provided the cross-section of the tank is large compared with that of the cylinder, we may consider that the cylinder is acted upon by a uniform field. We assume that the major axis of the elliptic cross-section is set at 45° to the walls of the tank and therefore to the direction of the applied field. The arrangement is illustrated in figure 1.

At the surface between the cylinder and fluid there exists a potential difference (phase-boundary potential) due to the presence of an electric double layer. A field acting upon the part of the double layer in the fluid produces the phenomenon of electrosmosis, and this in turn causes a general fluid motion in the tank. To determine the electrosmotic velocity, we have to know the part of the phase-boundary potential that is in the fluid (the electrokinetic potential), and the electric field strength acting upon the corresponding part of the double layer. We make the usual assumption that for a given pair of materials, the phase-boundary potential, and the above-mentioned part of it, are constants, and the field strength is obtained from the solution of the electrical boundary-value problem. In fact, as again is usual, we shall regard the double layer as very thin compared with the dimensions of the system, and we take the field at the solid boundary, on the side of the fluid. The reason for having all the vertical walls of the tank conducting is so that electrosmosis should not occur there but only at the cylinder, as will be explained in due course.

We shall calculate the fluid motion taking the electrosmotic velocity as a boundary condition at the cylinder, and treating the problem as one of quasi-steady viscous flow. The system is investigated in a region far from the top and bottom, where it can be regarded hydrodynamically, as it is electrically, as two-dimensional. The fluid motion is obtained, from the Stokes equations in terms of the stream function, which is found by numerical analysis. The stresses at the surface of the elliptic cylinder are then calculated, and hence the mechanical action on the body, of hydrodynamical origin, determined.

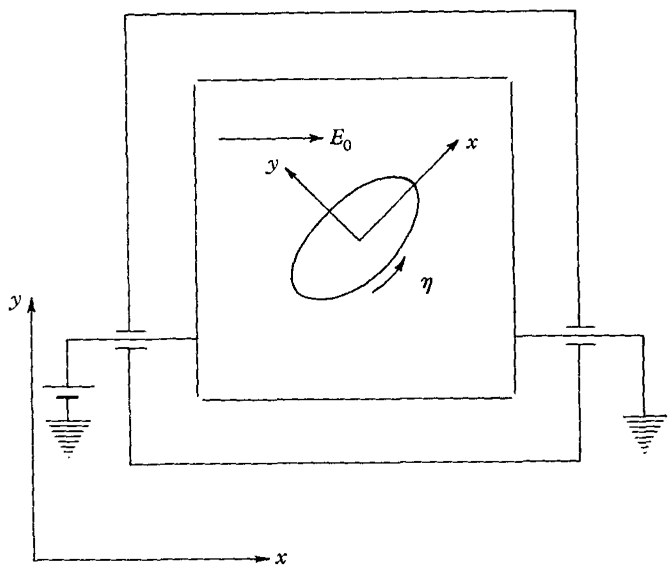

FIGURE 1. Cross-section of the system.

There is also a purely electrical mechanical action, but we assume that this can be calculated without considering the hydrodynamical problem.

Specific values have to be given to the dimensions of the system in order to apply the numerical method (this is one reason why the cylinder is set specifically at 45°) although, in fact, the results hold for all geometrically similar systems. The ratio of the dielectric constants (or the conductivities) of the cylinder and fluid must also be specified, but the calculation is done for three values of this ratio and graphs drawn so that our final results are in fact obtained as functions of the ratio.

We treat the electrical problem analytically, assuming that the field is the superposition of a part due to the cylinder and a perfectly uniform part corresponding to fixed sources at infinity. These conditions are very different from what we actually have and which are closely associated with what we use in the hydrodynamical problem. One might well ask whether the actual conditions are a good enough approximation to those assumed in the electrical problem, and if so, why one could not similarly assume hydrodynamically that the fluid extends to infinity, so that this part of the problem might also be tractable analytically. The answer to the first part of the question is that, in electrostatics, it is well known that a pair of parallel plates whose dimensions and separation are not so much greater than other dimensions of the system, gives a remarkably good approximation to the theoretical ideal, much better than one would be entitled to expect intuitively. Exact studies of edge-effects of condensors (e.g. Jeans 1943) show that a field becomes effectively uniform at quite a short distance inside the space between the plates, and a problem treated by King (1955), electrostatically very similar to ours except that the plates are infinite, shows strikingly that a body in the space between can be quite large in relation to the separation. King finds an error of less than 0·2 % through making a similar assumption to ours in a system with similar relative dimensions. Nevertheless, one might still reasonably propose that we could have treated the problem numerically, in the same way as the hydrodynamical problem, using the actual boundary conditions. However, there seems a considerable likelihood that the numerical procedure would have to be very highly refined to ensure that the error was no greater than we have through the approximation in the analytical method.

As regards the second part of the question, the problem of the electrosmotic flow round a fixed sphere in an infinite fluid was solved analytically by Khan (1958), who found that the velocity tends to a uniform non-zero value at infinity. We would expect the same behaviour in the solution of the corresponding problem of an elliptic cylinder. Now for a *finite* system, which is the only practicable kind, the velocity must become zero at the outer boundary of the system, and thus the solution here, illustrated in figure 3, representing two vortices in the fluid between the two boundaries, is of a kind we would expect. The point is this: the comparison between the finite and infinite problems is *qualitatively different* in the hydrodynamical case from what it is in the electrical case. Considering the plates which give the applied field, the field is roughly the same there in the infinite case with the plates absent. On the other hand, in the hydrodynamical case, the velocity field is quite different near the outer boundary from what it is if we remove the boundary and let the fluid extend to infinity. This situation, and the fact that the infinite problem is impracticable, is the justification of our problem. Of course, it is quite possible that the velocity field near the boundary of the ellipse is not very different in one case from the other, but in the absence of any analytical solutions of finite problems of this type, this is something that we cannot assume, and it may well be that the flow pattern is generally quite different.

2. The electrical problem

We define elliptic co-ordinates (ξ, η) by

$$\left.\begin{aligned} x &= c\cosh\xi\cos\eta, \\ y &= c\sinh\xi\sin\eta, \quad \xi \geqslant 0, \quad 0 \leqslant \eta < 2\pi. \end{aligned}\right\} \tag{1}$$

Laplace's equation in the two-dimensional system for a function V can be easily transformed to these co-ordinates and becomes

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}=0. \tag{2}$$

The general solution of equation (2) is

$$V=\sum_{n=0}^{\infty}(a_n \cosh n\xi+b_n \sinh n\xi)(c_n \cos n\eta+d_n \sin n\eta). \tag{3}$$

Given a uniform applied field of strength E_0 in the x-direction, we attempt to solve the steady electrical problem which is electrostatic if the cylinder and fluid are both insulators and steady-current if they are both conductors. We use ϵ to denote the ratio of the dielectric constant inside to that outside in the electrostatic case and the corresponding ratio of the conductivities in the steady-current case. We assume that all dielectric constants and conductivities are constant, so that the potential satisfies equation (2).

We consider the cross-section of the elliptic cylinder to be given by $\xi=\lambda$, a constant. When the field E_0 is applied such that its direction makes an anti-clockwise angle γ with the positive x-direction (the x-axis coinciding with the major axis of the elliptic section), we assume, according to (3), for the potential outside,

$$\begin{aligned}V=A&+\{(B-E_0\cos\gamma)\cosh\xi-B\sinh\xi\}c\cos\eta\\&+\{(C-E_0\sin\gamma)\sinh\xi-C\cosh\xi\}c\sin\eta,\end{aligned} \tag{4}$$

A, B and C being constants, and for the potential inside,

$$V'=-cB'\cosh\xi\cos\eta-cC'\sinh\xi\sin\eta, \tag{5}$$

B' and C' being constants and being equal to the x- and y-components, respectively, of a certain uniform field.

The boundary conditions are

$$V=V'+\Omega \quad (\xi=\lambda), \tag{6}$$

$$\left(\frac{\partial V}{\partial \xi}\right)_{\xi=\lambda}=\epsilon\left(\frac{\partial V'}{\partial \xi}\right)_{\xi=\lambda}, \tag{7}$$

where Ω is the phase-boundary potential which, according to what has been said before, we assume to be constant with respect to η. The values of the constants in (4) and (5) are determined with the use of (6) and (7).

Suppose that the fluid has a dielectric constant K and coefficient of viscosity μ, and that the electrokinetic potential at the elliptic boundary is ζ. Then we assume Smoluchowski's formula for the electrosmotic velocity,

$$v_t=\zeta K E_t/4\pi\mu \tag{8}$$

(e.g. Adam 1941), where v_t and E_t are the tangential components of velocity and electric intensity, respectively, at the boundary. We appreciate, of course, that v_t is actually a velocity at the fluid side of the double layer; however, E_t can be regarded as exactly at the boundary since with the assumption that the double layer is very thin, it does not vary appreciably through the thickness of the double layer. We have

$$E_t = -dV/ds = -dV'/ds, \tag{9}$$

where s is arc-length around the perimeter of the elliptic cross-section, being measured anticlockwise, corresponding to the direction of η increasing (see figure 1). Using the above solution (preferably V', being simpler) and properties of the elliptic co-ordinates, it follows that

$$E_t = \frac{E_0}{(\cosh^2\lambda - \cos^2\eta)^{\frac{1}{2}}}\left\{\frac{1+\tanh\lambda}{\epsilon+\tanh\lambda}\sinh\lambda\sin\gamma\cos\eta - \frac{1+\coth\lambda}{\epsilon+\coth\lambda}\cosh\lambda\cos\gamma\sin\eta\right\}. \tag{10}$$

We keep in mind our assumption that ζ is a constant with respect to η, being, like Ω, a characteristic of the two materials in contact.

At this point it is convenient to consider the mechanical action due purely to the electric field E_0. This is found by applying to the surface of the cylinder the Maxwell stress tensor, which, in Cartesian tensor notation, is

$$T_{\sigma\tau} = \frac{K}{4\pi}E_\sigma E_\tau - \frac{KE^2}{8\pi}\delta_{\sigma\tau} \quad (E^2 = E_\rho E_\rho,\ \rho,\ \text{etc.} = 1,2,3). \tag{11}$$

Substituting from the above solution, it is elementary, but tedious, to show that the cylinder experiences a couple per unit length of

$$G = \frac{1}{8}\frac{KE_0^2c^2(\epsilon-1)^2\sin 2\gamma}{(\epsilon+\coth\lambda)(\epsilon+\tanh\lambda)}, \tag{12}$$

and no total force.

3. The hydrodynamical problem

We choose new x- and y-axes, so that the walls of the tank are parallel to one or other of these axes (see figure 1).

We assume that the fluid is incompressible and, corresponding with its electrical properties, uniform. We then consider that the velocity (v_x, v_y) and pressure p in the body of the fluid satisfy the Stokes equations for quasi-steady viscous flow in the horizontal xy-plane, with no external force:

$$\left.\begin{aligned}\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)-\frac{\partial p}{\partial x} &= 0,\\ \mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right)-\frac{\partial p}{\partial y} &= 0.\end{aligned}\right\} \tag{13}$$

Together with these we take the equation of continuity,

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y} = 0. \tag{14}$$

Through (14), we introduce the stream function ψ, such that

$$v_x = -\frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \psi}{\partial x}, \tag{15}$$

and which, according to (13), satisfies the biharmonic equation,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi = 0. \tag{16}$$

In viscous flow at fixed solid surfaces the boundary conditions on the velocity are

$$v_n = 0, \quad v_t = 0, \tag{17}$$

the suffixes denoting normal and tangential components, respectively. Consider the outer boundary in our problem. It consists mainly of the four vertical walls of the tank, separated by small gaps. We assume that the gaps are solid and rounded, so that the boundary is a continuous smooth curve. Under these conditions, the first of equations (17) leads through (15) to the fact that ψ has a constant value, Γ_0 say, at the boundary. Then, since the walls of the tank are parallel to the x- or y-direction, it follows by the further use of (15) that equations (17) are, at the walls, replaced by the conditions on the stream function,

$$\psi = \Gamma_0 \quad \left\{\begin{array}{l}\partial\psi/\partial x = 0 \text{ (wall in } y\text{-direction),}\\ \partial\psi/\partial y = 0 \text{ (wall in } x\text{-direction.}\end{array}\right\} \tag{18}$$

Under electrostatic conditions there is no appreciable electrosmosis at this boundary. The walls are conductors, so that the electric intensity is normal and E_t in Smoluchowski's formula (8) is zero, and we assume that the effects of the insulating corners are negligible in this respect. Although the argument applies to the electrostatic situation, it holds to a good enough approximation in the steady-current case if, as we stipulate, the walls are much better conductors than the fluid and cylinder. This is physically realistic, for usually the walls would be made of metal, and metals have conductivities of a higher order than most fluids.

Conditions (17) apply also at the elliptic boundary, but electrosmosis occurs there, whence there is an abrupt change of tangential velocity on going into the fluid. Thus v_t when it means the *precise* tangential velocity, is given by (17), but when it means the tangential velocity at the fluid side of the thin double layer, it is given by (8). Our procedure will be to take v_t as given by (8) as a boundary condition; we thus obtain an account of the fluid motion which is inaccurate to the extent that our value of v_t does not occur exactly at the elliptic boundary and incomplete in that there is no description of the motion in the region occupied by the double layer. Under the assumption of a very thin double layer, we regard this deficiency as unimportant. Our boundary conditions for the ellipse are now

$$v_n = 0, \quad v_t = QX, \tag{19}$$

where Q and X, introduced for convenience in the numerical work, are given by

$$Q = \zeta K E_0 \sin\gamma/4\pi\mu, \quad X = E_t/E_0 \sin\gamma. \tag{20}$$

With the use of (15), these conditions are shown to be equivalent to the conditions on the stream function,

$$\psi = \Gamma_i, \quad -\frac{\partial\psi}{\partial y}\cos\alpha + \frac{\partial\psi}{\partial x}\sin\alpha = QX, \tag{21}$$

where Γ_i is a new constant and α is the angle the tangent to the ellipse makes with the x-direction, measured anticlockwise, the direction of the tangent being that of the arc-length s increasing.

We assume that under the boundary conditions (17) and (19), the biharmonic equation (16) has a solution which is unique except for an additive constant. It follows that the solution is determined completely if we give an arbitrary value to one of the constants Γ_0 and Γ_i (the other being then fixed automatically). We shall take

$$\Gamma_0 = 0. \tag{22}$$

Suppose that we reverse the direction of the applied field E_0, and require that the new stream function satisfy (18) as restricted by (22). It follows from the electrical problem, equations (8) and (19), and the considerations of uniqueness, that the new ψ is minus the old one. Thus Γ_i is changed to $-\Gamma_i$. Let us now rotate the whole system through 180° in the xy-plane, the co-ordinate axes remaining fixed. Then ψ is unchanged, being a pseudo-scalar, and we see that the system has become as it was originally, before the field was reversed. In this new manifestation of the original system, ψ is zero at the outer boundary, as before, but is equal to $-\Gamma_i$ instead of Γ_i at the inner boundary. It follows from uniqueness that

$$\Gamma_i = 0. \tag{23}$$

4. The numerical method: calculation of the stream function

The square boundary of the system is chosen to be of side 40 cm. For the ellipse with major and minor axes $2a$ and $2b$, respectively, eccentricity e and distance between the foci, $2c$, we take

$$2a = 19{\cdot}88\,\text{cm}, \quad 2b = 9{\cdot}94\,\text{cm},\dagger \tag{24}$$

so that $e = 0{\cdot}866, \quad 2c = 17{\cdot}22\,\text{cm}, \quad \cosh\lambda = 1{\cdot}1547, \quad \sinh\lambda = 0{\cdot}5773.$ (25)

Four values of γ less than 360° (all of course equivalent) will make the major axis of the elliptic section inclined at 45° to the direction of E_0, and we take $\gamma = 135°$, giving the configuration shown in figure 1.

We determine the stream function ψ at discrete values of x and y forming a grid of square mesh in the xy-plane, in the region between the two boundaries. The x- and y-co-ordinates of any point are given by

$$(x, y) = (x_0 + jh, y_0 + kh), \tag{26}$$

where j and k are integers, h is the mesh-length and (x_0, y_0) is an arbitrary starting-point. For simplicity the point in equation (26) is referred to as (j, k) and functions at the point by corresponding suffixes.

The finite-difference approximation to the biharmonic equation, in accordance with the derivation and notation given by Allen (1954), is

$$\frac{1}{h^4}\left|\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -8 & 2 & 0 \\ 1 & -8 & 20 & -8 & 1 \\ 0 & 2 & -8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right| \psi_{j,k} = 0, \tag{27}$$

† These awkward values arose through deviations from the intended values of 20 and 10 cm which occurred when an accurate drawing was made.

the centre number referring to the point (j, k). The numbers also indicate changes in the residuals at the corresponding points for unit change in the value of the function at (j, k).

We have to suitably adapt the boundary conditions to the finite-difference method. Taking the outer boundary and, say, the left-hand wall of the tank in the y-direction (figure 1), let us for the sake of argument call a point on this part of the boundary, $(j-1, k)$. Then from (18), (22) and an approximation for the derivative of a function with respect to x, we find that $\psi_{j-1,k} = \psi_{j,k} = 0$. In this way, we see immediately that ψ is zero all along the two outermost layers of mesh-points.

As regards the inner boundary, we have, solving (21) for the derivatives of ψ and using (23),

$$\psi = 0, \quad \frac{\partial \psi}{\partial x} = QX \sin\alpha, \quad \frac{\partial \psi}{\partial y} = -QX\cos\alpha. \tag{28}$$

We need to determine the values of the function at points such as $(j-1, k)$ and $(j-2, k)$ in figure 2, termed the selvage and fictitious points. The method is almost the same as that given by Allen (1945). We expand ψ into a Taylor series in the neighbourhood of the points of intersection and eliminate terms involving derivatives higher than first-order. Applying the boundary conditions (28), we obtain

$$\left.\begin{aligned} \psi_{j\mp1,k} &= \pm\frac{hd}{h+d}QX\sin\alpha + \frac{d^2}{(h+d)^2}\psi_{j,k}, \\ \psi_{j\mp2,k} &= \mp\frac{2h(h-d)}{h+d}QX\sin\alpha + \left(\frac{h-d}{h+d}\right)^2\psi_{j,k}, \end{aligned}\right\} \tag{29}$$

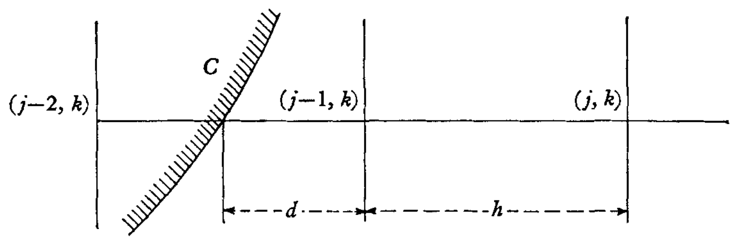

FIGURE 2. Intersection of a mesh-line with the ellipse.

where the minus and plus signs in the suffixes on the left refer, respectively, to points in the fluid on the right and on the left of the ellipse as shown in figure 1 (only the right side being illustrated in figure 2). Similarly, where a mesh-line parallel to the y-direction intersects the ellipse,

$$\left.\begin{aligned} \psi_{j,k\pm1} &= \pm\frac{hd}{h+d}QX\cos\alpha + \frac{d^2}{(h+d)^2}\psi_{j,k}, \\ \psi_{j,k\pm2} &= \mp\frac{2h(h-d)}{h+d}QX\cos\alpha + \left(\frac{h-d}{h+d}\right)^2\psi_{j,k}, \end{aligned}\right\} \tag{30}$$

the plus and minus signs in the suffixes on the left referring, respectively, to points in the fluid on the lower and upper sides of the ellipse (figure 1). In each case (j, k) is the second point out from the elliptic boundary and d is the distance between the point of intersection and the selvage point.

Taking $\epsilon = 1$ in the first place, a mesh was chosen with $h = 4$ cm. The value of Q was taken to be 1, for the results depend linearly upon Q and we can therefore multiply our final results by the actual value of this quantity. At each point of intersection of the ellipse and a mesh-line, the value of X was calculated, and the results were tabulated. Values of d and α were measured from an accurate drawing and, assigning arbitrary values to the $\psi_{j,k}$ in equations (29) and (30),

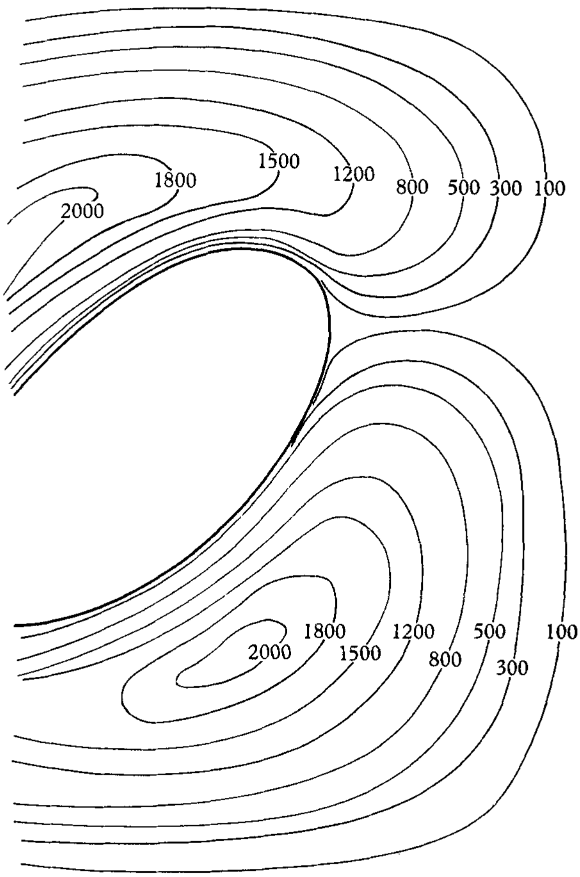

FIGURE 3. Streamlines and associated values of $\psi \times 10^3$ for $\epsilon = 1$.

the values of the function at the selvage and fictitious points were calculated. To avoid the use of decimals, the ψ-values were multiplied by 100. Arbitrary values of 100ψ were guessed for the other mesh-points except the two outermost layers, the residuals were calculated and the relaxation process carried out. When the residuals were effectively zero, the work was repeated on a finer mesh for which $h = 2$ cm; finally, it was done on the finer mesh for values of 1000ψ until the residuals were reduced to the least possible values. There was seen to be antisymmetry about any line through the centre of the ellipse; the final function values are given by Rowe (1962) and we do not quote them in detail, but the plot enabled streamlines to be drawn which are shown in figure 3. The whole process was repeated for $\epsilon = 3, 5$. Again the detailed results are given by Rowe (1962).

5. Calculation of the hydrodynamical stresses and the force

The components of the hydrodynamical stress tensor acting in a region of uniform, viscous, incompressible fluid are given in terms of the stream function by

$$\left.\begin{aligned} T_{xx} &= -p-2\mu\frac{\partial^2\psi}{\partial x\,\partial y},\\ T_{yy} &= -p+2\mu\frac{\partial^2\psi}{\partial x\,\partial y},\\ T_{xy} &= T_{yx} = \mu\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right).\end{aligned}\right\} \tag{31}$$

Using the ψ-values determined from the relaxation method, the values of $\partial^2\psi/\partial x^2$ and $\partial\psi^2/\partial y^2$ were calculated from the finite-difference approximations (Allen 1954) at points such as (j,k) and $(j-1,k)$ in figure 2, and by linear extrapolation, the values at the points of intersection were found.

A special technique is used to find the finite-difference approximations for $\partial^2\psi/\partial x\,\partial y$, and is described in detail by Rowe (1962).† In considering points of intersection of mesh-lines in the x-direction and the ellipse, the essence of the method is the determination of $\partial\psi/\partial y$ at the points (j,k) and $(j-1,k)$ (the ψ-value at a fictitious point being used in the case of $(j-1,k)$) from the finite-difference approximation (Allen 1954) (see figure 2). From the boundary conditions (28), the value of $\partial\psi/\partial y$ is known at C in figure 2. Using the values of $\partial\psi/\partial y$ at the above-mentioned points, the values of $\partial^2\psi/\partial x\,\partial y$ at $(j-\frac{1}{2},k)$ (the notation is self-explanatory), and at a point half-way between C and $(j-1,k)$, can be found by finite-difference differentation in the x-direction. By linear extrapolation in this direction, the values of $\partial^2\psi/\partial x\,\partial y$ at C and similar points were determined. For points of intersection on mesh-lines in the y-direction, the method is basically the same, but initially the values of $\partial\psi/\partial x$ are calculated and extrapolation is in the y-direction.

From equations (13) and (15), we have

$$\frac{\partial p}{\partial x} = -\mu\frac{\partial}{\partial y}(\nabla^2\psi), \quad \frac{\partial p}{\partial y} = \mu\frac{\partial}{\partial x}(\nabla^2\psi). \tag{32}$$

Integrating with respect to arc-length on the ellipse,

$$p-p_0 = \int_0^s \frac{dp}{ds}\,ds, \tag{33}$$

where p_0 represents the pressure at an arbitrary point from which s is measured. Since

$$\frac{dp}{ds} = \frac{\partial p}{\partial x}\frac{dx}{ds}+\frac{\partial p}{\partial y}\frac{dy}{ds}, \tag{34}$$

then, from (34), (33) and (32) we see that

$$p = p_0+\mu\int_0^s\left\{-\frac{\partial}{\partial y}(\nabla^2\psi)\cos\alpha+\frac{\partial}{\partial x}(\nabla^2\psi)\sin\alpha\right\}ds. \tag{35}$$

† It has been pointed out by a referee that we could instead have used a method such as was used for ψ and led to equations (29) and (30). It might have been better on grounds of general consistency to have done so.

We shall take $\mu = 1$, for as with Q, the results depend linearly upon it and we can multiply by μ at the end.

The following method is used to find the finite-difference approximations to the values of $\partial(\nabla^2\psi)/\partial x$ and $\partial(\nabla^2\psi)/\partial y$, and is described fully by Rowe (1962). To find $\partial(\nabla^2\psi)/\partial x$ at C say, in figure 2, we use the known values of $\partial^2\psi/\partial x^2$, $\partial^2\psi/\partial y^2$, and consequently $\nabla^2\psi$, at $(j-1,k)$, (j,k) $(j+1,k)$ to calculate $\partial(\nabla^2\psi)/\partial x$ at $(j-\frac{1}{2},k)$ and $(j+\frac{1}{2},k)$ in the usual way, and by linear extrapolation in the x-direction, we find $\partial(\nabla^2\psi)/\partial x$ at C. For a point on a mesh-line in the y-direction and on the lower side of the ellipse, we use the fact that $\partial^2\psi/\partial x^2$ and $\partial^2\psi/\partial y^2$, and consequently $\nabla^2\psi$, at the points of intersection had been determined by linear extrapolation, so that $[\partial(\nabla^2\psi)/\partial x]_{j-1,k} = [\partial(\nabla^2\psi)/\partial x]_{j-\frac{1}{2},k}$. Using the values of $\nabla^2\psi$ at $(j-2,k-1)$ and $(j,k-1)$ and the finite-difference procedure, we obtain $\partial(\nabla^2\psi)/\partial x$ at $(j-1,k-1)$. By linear extrapolation in the y-direction we obtain $\partial(\nabla^2\psi)/\partial x$ at the point of intersection. Essentially the same method applies to the determination of this quantity at points on the upper side of the ellipse. In order to get the values of $\partial(\nabla^2\psi)/\partial y$, a reverse type of procedure holds; for instance in the case of C, we obtain $[\partial(\nabla^2\psi)/\partial y]_{j-1,k}$ and $[\partial(\nabla^2\psi)/\partial y]_{j,k}$, and extrapolate in the x-direction.

The arc-lengths δs between the points of intersection were measured from the accurate drawing. Using these values, values of the appropriate derivatives of ψ, and the measured values of α, the trapezoidal rule was applied to find approximate values of the integral in (35), so that, remembering the remark following that equation, $p-p_0$ was found at the points of intersection.

The components of force per unit length on the cylinder are given by

$$\left.\begin{aligned} F_x &= \oint (T_{xx}\sin\alpha - T_{xy}\cos\alpha)\,ds, \\ F_y &= \oint (T_{xy}\sin\alpha - T_{yy}\cos\alpha)\,ds. \end{aligned}\right\} \tag{36}$$

We calculate these quantities using the trapezoidal rule again, under the conditions already stated that $Q = 1$, $\mu = 1$, and taking p_0 as zero, as we may since it does not contribute anything. Denoting F_x, F_y for these values of Q and μ by Φ_x, Φ_y, respectively, the final results for $\epsilon = 1, 3$ and 5 are given in table 1. They are also recorded graphically in figure 4, and this graph gives functions $\Phi_x(\epsilon)$, $\Phi_y(\epsilon)$ from which values for any ϵ-value from 1 to 8 can be read. Then introducing Q and μ, and substituting the value of Q from (20) (with $\gamma = 135°$), we obtain

$$F_x = \frac{\zeta K E_0}{4\pi\sqrt{2}}\Phi_x(\epsilon), \quad F_y = \frac{\zeta K E_0}{4\pi\sqrt{2}}\Phi_y(\epsilon). \tag{37}$$

	$\epsilon = 1$	$\epsilon = 3$	$\epsilon = 5$
Φ_x	$-24{\cdot}71$	$-13{\cdot}04$	$-9{\cdot}089$
Φ_y	$-4{\cdot}753$	$-4{\cdot}834$	$-3{\cdot}947$

TABLE 1

By considering how quantities change under a linear magnification of the system, we find that these results hold for all geometrically similar systems.

An examination of the detailed numerical results shows easily without doing the calculation, that the cylinder experiences no couple due to the hydrodynamical stresses, as can also be predicted from first principles.

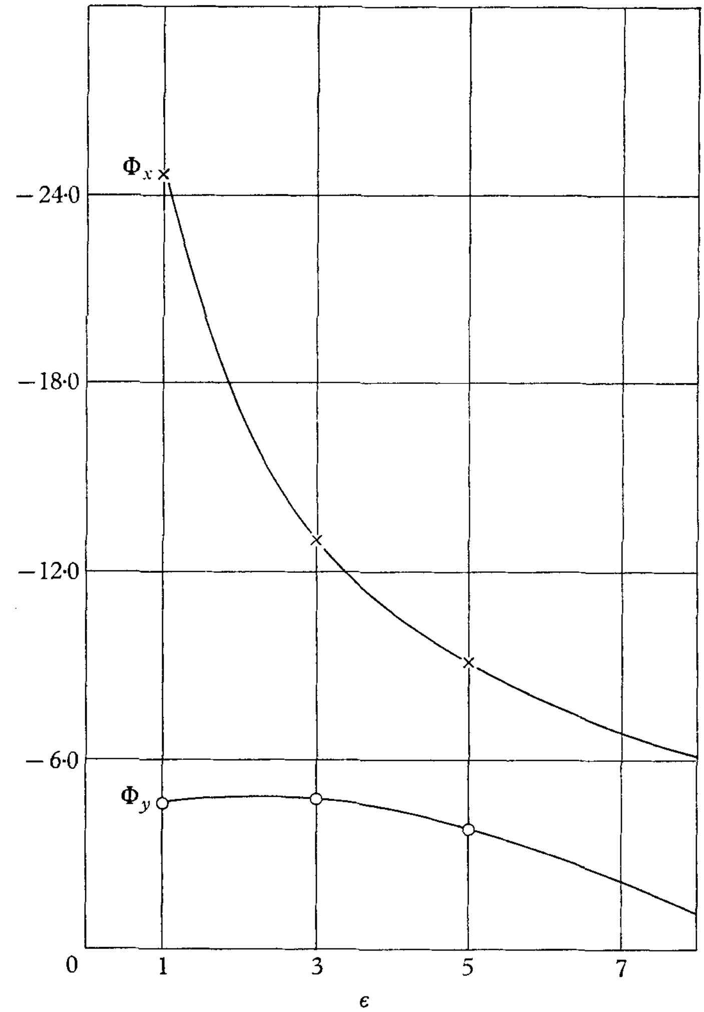

FIGURE 4. The functions $\Phi_x(\epsilon)$, $\Phi_y(\epsilon)$.

6. Possible experimental application

The above results could be used as the basis of an experimental method for determining electrokinetic potentials. The method would essentially be to determine the force on an elliptic cylinder under the conditions described and then calculate ζ by substitution in one or other of equations (37). This could be done by measuring the couple on two exactly similar cylinders hanging from a uniform beam, with equal lengths immersed in fluid in exactly similar tanks. The fields in the two systems could be both perpendicular or both parallel to the beam, and in order to produce a couple, they would have to be in opposite directions; in the former case we get a couple due to forces of the type F_x in (37), and in the latter case, due to forces of the type F_y. The major axes of the elliptic cross-sections must, of course, be at 45° to the beam; they must also be at right angles to each other, for we bear in mind that there are purely electrical couples as given by (12), and with this arrangement, these couples cancel. The beam would be suspended from a torsion head, and the couple measured in the usual way by turning the torsion head to restore a rotation of the beam.

If a length l of each cylinder is immersed, we cannot of course take the force on the cylinder as given in terms of (37) by lF_x or lF_y, because the system cannot be regarded as two-dimensional near the top and bottom, and (37) will not be valid there. However, the end-effects can be eliminated by doing two experiments with two different values of l, l_1 and l_2 say. Under these conditions, (37) is applicable by giving the *difference* of the force on each cylinder as $|l_2 - l_1| F_x$ or $|l_2 - l_1| F_y$.

It is desirable on general grounds that one should determine both F_x and F_y, but there is the special point in this context that the errors inherent in the numerical method enter in different ways in the calculation of $\Phi_x(\epsilon)$ and $\Phi_y(\epsilon)$. The stress T_{xx} is used to calculate F_x, and T_{yy} to calculate F_y, and from formulae (31), it can be seen that in one of the quantities T_{xx} and T_{yy}, the errors in p and $\partial^2\psi/\partial x\,\partial y$ reinforce, and in the other they tend to cancel.

7. Discussion

The quantitative correctness of our results will of course depend on the validity of Smoluchowski's formula. The usual derivation is rather crude, depending on the 'parallel-plate' concept of the double layer. Henry (1931) has shown that it is correct under certain assumptions concerning the double layer and for certain simple surfaces, with no restriction on the double-layer thickness. Cade (1954) has shown that under simple assumptions concerning the double-layer structure, it is true for any smooth surface provided the double layer is thin. However, even if the formula is not correct, it seems very likely that the relationship between v_t and E_t will be one of proportionality, so that our results are probably correct except for a constant factor.

It is remarkable that the general results (37) do not depend on viscosity, although the essence of the hydrodynamical method is the equation of motion of viscous flow. The stresses are effectively proportional to viscosity, but the electrosmotic velocity is inversely proportional, and the effects cancel.

The theory assumes that it is correct to superpose linearly the hydrodynamical stresses upon the electrical stresses given according to classical electrostatics. Brown (1951) worked out the general theory of the stress system in a static fluid influenced by an electrostatic field, but apart from the work of Cade (1954), concerning boundary effects, nothing comparable seems to have been done for a moving viscous fluid. However, precisely the same assumption was made by Henry (1931), and probably by all workers in the subject.

The ellipse is chosen to be of small eccentricity. This is done since there would be a large curvature at the ends otherwise, and the approximations would be unreliable there. This is the more serious because it is likely that with large curvature there would be large stresses, as in the situation leading to the Cisotti paradox (see Birkhoff 1955), so that the largest errors would occur in the most important contributions to the final result.

This paper is based upon a thesis by A. R., of the same title, accepted by the University of London in partial fulfilment of the requirements for the M.Sc. degree.

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